

Footnotes to a Paper of Domokos, I

Allan Berele
 Department of Mathematics*
 DePaul University
 Chicago, IL 60614

January 15, 2015

Abstract

We use Regev's double centralizer theorem from [8] to study invariants of matrices with group actions. This theory then has applications to trace identities and Poincaré series.

Keywords: Trace identities, invariant theory, Poincaré series, double centralizer

Although our constructions will work over any field, the double centralizer theorem requires characteristic zero and we will take F to be a characteristic zero field throughout.

In [8] Regev proved the following double centralizer theorem. Let V be a finite dimensional vector space over a characteristic zero field F , let A be a semisimple subalgebra of $GL(V)$, and let G be the group of units in $End_A(V)$. Then Regev proved that G and the wreath product $A \sim S_k$ have the double centralizer property in $End(V^{\otimes n})$. Knowing how the classical Schur-Weyl duality had been used to study invariant theory and trace identities, Domokos applied Regev's theory to study the invariant theory of matrices broken into rectangular blocks. These correspond to the case of $V = V_1 \oplus \cdots \oplus V_t$ and $G = GL(V_1) \times \cdots \times GL(V_t)$, and Domokos explored the invariant theory and the trace identities corresponding to G . We will describe Domokos's theory in more detail in the course of this paper.

Although one could hardly fault Domokos for choosing the most natural and important applications of Regev's theory to investigate, there are a number of applications he didn't consider we believe worth exploring. These include

*Partially supported by a Faculty Research Grant from DePaul University

- Computing the Poincaré series of the universal algebras, especially using complex integrals; and computing the pure and mixed trace cocharacters.
- Generalizing Procesi's embedding theorem from [7].
- Dealing with other $G \subseteq GL(V)$.

Before jumping into the body of the paper we give an example of the last two application, which seems especially novel. Let $GL_2(F)$ embed into $M_6(F)$ via

$$a \longrightarrow \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}.$$

Let $A \cong M_3(F)$ be the commuting ring. We form the free A -algebra $A\langle X \rangle$ in which elements of X are assumed to commute with elements of F but not with elements of A . This would be the set of A -polynomials and we could ask about A -identities for $M_6(F)$. However, our interest will be in the pure or mixed trace identities. We enlarge $A\langle X \rangle$ to an algebra with trace in the natural way to form A -trace polynomials, and get these two theorems.

Theorem. *Let $\Phi : M_6(F)^n \rightarrow F$ be invariant under simultaneous conjugation from $GL_2(F)$, considered as a subring of $M_6(F)$ as above. Then Φ is equal to a pure trace $M_3(F)$ polynomial. This means that the algebra of invariant maps is generated by the maps*

$$(x_1, \dots, x_n) \longrightarrow tr(a_1 x_{i_1} \cdots a_k x_{i_k}),$$

where $1 \leq i_\alpha \leq n$ and the $a_i \in A$. Likewise, the mixed trace $M_3(F)$ polynomials give all invariants functions from $M_6(F)^n$ to $M_6(F)$.

Theorem. *Let f_i , $i = 1, 2, 3$ be image of the matrix unit $e_{ii} \in M_3(F) \equiv A$ in $M_6(F)$, so that $f_i M_6(F) f_i$ is isomorphic to $M_2(F)$; and let CH be the degree two Cayley-Hamilton function, so that CH will be an identity for each $f_i M_6(F) f_i$. Then all A trace identities of $M_6(F)$ will be consequences of these three.*

Theorem. *Let S be an algebra with A -action and trace which satisfies the A -trace identities of the previous theorem. Then S can be embedded into 6×6 matrices over a commutative ring, and this embedding will preserve both A -action and trace.*

In a subsequent paper we hope to explore the \mathbb{Z}_2 -graded analogues of the topics in [3] and the current paper.

1 E -Polynomials

The first case we consider, following Domokos, will be algebras R with 1, in which $1 = e_1 + \cdots + e_t$, a specified sum of orthogonal idempotents. Let $F\langle X, E \rangle$ be the free algebra on the set $X \cup E$ where X is a set of indeterminates and E is the set $\{E_1, \dots, E_t\}$ modulo the relations that the E_i are orthogonal idempotents summing to 1. Note that $F\langle X, E \rangle$ is spanned by the monomials

$$E_{i_1} X_{j_1} E_{i_2} X_{j_2} \cdots X_{j_s} E_{j_{s+1}}.$$

This algebra has the universal property that given any algebra R with t orthogonal idempotents summing to 1, as above, and given any set theoretic map $f : X \rightarrow R$ there is a unique extension of f to a homomorphism $F\langle X, E \rangle \rightarrow R$ taking each E_i to e_i . Considering the elements of $F\langle X, E \rangle$ to be E -polynomials, we have the natural concepts of E -polynomial identities. For example, if $R = M_2(F)$, $e_1 = e_{11}$ and $e_2 = e_{22}$ then the following would be E -polynomial identities:

$$E_1 X_1 E_1 X_2 E_1 = E_1 X_2 E_1 X_1 E_1, \quad E_1 X_1 E_2 X_2 E_1 = E_2 X_2 E_1 X_1 E_2.$$

As usual in p.i. theory the set of identities of an algebra R is an ideal $Id(R)$ in $F\langle X, E \rangle$, and the quotient is a universal algebra $U(R)$. In the example of 2×2 matrices we have been considering, the universal algebra can be thought of as either the algebra generated by the ordinary generic matrix together with e_{11} and e_{22} ; or as the algebra generated by all $e_{ii} X_\alpha e_{jj}$:

$$\begin{pmatrix} x_{11}^{(\alpha)} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_{12}^{(\alpha)} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ x_{21}^{(\alpha)} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & x_{22}^{(\alpha)} \end{pmatrix}$$

together with e_{11} and e_{22} . We point out here that our setting is slightly different from Domokos's. He did not use the language of E -identities and his generic algebra would have included the $e_{ii} X_\alpha e_{jj}$ but not the idempotents e_{ii} themselves.

In this paper our main interest will be in trace identities rather than identities. In the standard manner we generalize $F\langle X, E \rangle$ to $F\langle X, E, tr \rangle$ the generic algebra with orthogonal idempotents and trace. We note that, because of the orthogonality of the idempotents, in order for the trace of a monomial $E_{i_1} x_{j_1} \cdots x_{j_m} E_{i_{m+1}}$ to be non-zero we must have $i_1 = i_{m+1}$ and the trace of the monomial will be

$$tr(E_{i_1} x_{j_1} \cdots x_{j_m} E_{i_1}).$$

It will be convenient to assume that we are given positive integers n_i and that the universal algebra satisfies the identities $tr(E_i) = n_i$ for all i and so $tr(1) = \sum n_i$. Hence, every trace term can be assumed to have at least one x_j in it.

The ideal of trace identities of R will be denoted $Id(R, tr)$, the ideal of pure trace identities will be denoted $Id(R, ptr)$, and the corresponding generic algebras as $U(R, tr)$ and $U(R, ptr)$. In the case of matrices, $U(M_n(F), ptr)$ will be the subalgebra of the polynomial ring $F[x_{ij}^{(\alpha)}]_{ij\alpha}$ generated by the traces of the elements of $U(M_n(F))$,

$$tr(e_{i_1} X_{j_1} \cdots e_{i_m} X_{j_m}).$$

(The X are now capitalized because they are generic matrices, while the e are lower case to denote specific idempotents in $M_n(F)$.) And, as in the classical case, $U(M_n(F), tr)$ will be the algebra generated by $U(M_n(F))$ and $U(M_n(F), ptr)$.

2 Theorems of Regev and Domokos

Regev's work in [8] generalized the classic Schur-Weyl duality. Let V be an n dimensional vector space and let W be the tensor power $V^{\otimes k}$. Then W is a module for the symmetric group via

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

and the general linear group acts on W by the diagonal action

$$g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k.$$

It is not hard to see that these two actions commute with each other. It is less easy to see, but a theorem with applications in many areas, that each spans the commutator of the other in $End(W)$. In Regev's generalization, we start off with a group G that is a finite direct product of matrix groups

$$G = GL_{n_1} \times \cdots \times GL_{n_t}$$

that acts on V . Let A be the commutator algebra of G in $End(V)$. The group G and the wreath product $A \sim S_k$ act on $W = V^{\otimes k}$. Regev proved the double centralizer theorem for these actions. Moreover, the algebra $A \sim S_k$ is semisimple and so decomposes into a sum of simple two-sided ideals $\oplus I_{\langle \lambda \rangle}$ indexed by t -tuples of partitions, $\langle \lambda \rangle = (\lambda_1, \dots, \lambda_t)$; and $I_{\langle \lambda \rangle}$ acts as zero on W precisely when some λ_i has height greater than n_i .

In [3] Domokos applied Regev's theorem to the trace identities of matrices much as Procesi had applied Schur-Weyl. He restricted his attention to the case in which $G = GL(V_1) \oplus \cdots \oplus GL(V_t)$ acts in the obvious way on $V = V_1 \oplus \cdots \oplus V_t$. In this case the centralizer algebra A is t copies of the field, $A = F^t$. Let $1 \in A \subseteq \text{End}(V)$ decompose as $e_1 + \cdots + e_t$, where the e_i are orthogonal idempotents and each e_i has rank equal to the dimension of V_i . Here are Domokos' analogues of the classic theorems of matrix invariants:

Theorem 2.1. *The pure E -trace polynomials give all functions from $M_n(F)^d$ to the field F invariant under simultaneous conjugation from G .*

In the classical case the group $GL_n(F)$ has an action on the polynomial algebra $F[x_{ij}^{(\alpha)}]$ defined by $g(x_{ij}^{(\alpha)})$ equals the (i, j) entry of $gX_\alpha g^{-1}$. In the current case we restrict this action to $G \subseteq GL_n(F)$.

Theorem 2.2. *The pure trace ring $U(M_n(F), \text{ptr})$, which is the ring generated by the traces of elements of $U(M_n(F))$, equals the fixed ring of G acting on the polynomial algebra $F[x_{ij}^{(\alpha)}]$.*

A virtue of the language of E -polynomials is that makes it easy to pass from pure trace identities to mixed ones. Let $f(x_1, \dots, x_{k+1})$ be a multilinear pure trace E -polynomial. The f can be written in the form

$$f = \sum_{\alpha} g_{\alpha} \text{tr}(u_{\alpha} x_{k+1}) = \text{tr}(\sum_{\alpha} g_{\alpha} u_{\alpha} x_{k+1}),$$

where each g_{α} is a pure trace E -polynomial and each u_{α} is an E -polynomial, and so $\sum g_{\alpha} u_{\alpha}$ is a multilinear, mixed trace E -polynomial in x_1, \dots, x_k . Moreover, since the trace is non-degenerate we get the following corollaries to Theorems 2.1 and 2.2.

Corollary 2.3. *The mixed E -trace polynomials give all functions from $M_n(F)^d$ to $M_n(F)$ invariant under simultaneous conjugation from G .*

The group $GL_n(F)$, and by restriction the group G , acts on $n \times n$ matrices over the polynomial ring $F[x_{ij}^{(\alpha)}]$. We consider

$$M_n(F[x_{ij}^{(\alpha)}]) = F[x_{ij}^{(\alpha)}] \otimes M_n(F).$$

The action of $GL_n(F)$ on the first factor is as above; and the action on the second factor is via $g(a) = g^{-1}ag$.

Corollary 2.4. *The (mixed) trace ring $U(M_n(F), tr)$, which is the ring generated by $U(M_n(F))$ together with the pure trace ring $U(M_n(F), ptr)$, equals the fixed ring of G acting on the matrix algebra $M_n(F[x_{ij}^{(\alpha)}])$.*

Finally, Domokos also generalizes the Razmyslov-Procesi theorem. His result is based on the multilinear Cayley-Hamilton identity, $CH_n(x_1, \dots, x_n)$, a pure trace polynomial equivalent to the usual Cayley-Hamilton, see [6].

Theorem 2.5. *The ideal of pure E -trace identities for $M_n(F)$ is generated by the Cayley-Hamilton identities restricted to each diagonal component*

$$CH_{n_i+1}(e_{ii}X_1e_{ii}, \dots, e_{ii}X_{n_i+1}e_{ii})$$

Now that we have extended Domokos's results to mixed traces, we can easily extend this one also. Define the mixed trace polynomial $MCH_n(x_1, \dots, x_{n-1})$ via

$$tr(MCH_n(x_1, \dots, x_{n-1})x_n) = CH_n(x_1, \dots, x_n).$$

Then by the non-degeneracy of the trace we have

Theorem 2.6. *The ideal of mixed E -trace identities for $M_n(F)$ is generated by the mixed Cayley-Hamilton identities restricted to each diagonal component*

$$MCH_{n_i}(e_{ii}X_1e_{ii}, \dots, e_{ii}X_{n_i+1}e_{ii})$$

3 Cocharacters and Poincaré Series

The free algebras $F\langle X, E \rangle$, $F\langle X, E, tr \rangle$ and $F\langle X, E, ptr \rangle$ each have an m -fold grading with $m = |E|^2 \times |X|$ in which $e_i x_\alpha e_j$ has grading $(0, \dots, 0, 1, 0, \dots)$, where the 1 is in the position corresponding to (i, j, α) . So, for example, the degree of

$$tr(e_{11}X_3e_{22}X_2e_{44}X_1) = tr(e_{11}X_3e_{22}e_{22}X_2e_{44}e_{44}X_1e_{11})$$

would be 1 in each of the degrees $(1, 2, 3)$, $(2, 4, 2)$ and $(4, 1, 1)$. The ideals of identities of algebras are homogenous ideals and so the quotients, which are the universal algebras, are also graded. This implies that they will have Poincaré series in m variables which we may call $t(i, j, \alpha)$. For fixed i, j the

Poincaré series will be symmetric in the $t(i, j, \alpha)$ and so the Poincaré series can be written as

$$\sum_{i,j=1}^{|E|} \sum_{n_{ij}=0}^{\infty} \sum_{\lambda_{ij} \vdash n_{ij}} m(\lambda_{11}, \lambda_{12}, \dots, \lambda_{|E|, |E|}) S_{\lambda_{11}}(t(1, 1, \alpha)) S_{\lambda_{12}}(t(1, 2, \alpha)) \dots$$

Note that the coefficients $m(\lambda)$ depends only on the algebra R and the partitions and not on the number of variables, except to the extent that the Schur function $S_{\lambda_{ij}}$ will be zero unless the number of variables $t(i, j, \alpha)$ is greater than or equal to the height of the partition λ_{ij} .

We work first with the Poincaré series for the pure trace ring which we will denote $P(t, ptr)$ or $P(t, ptr; n_{11}, \dots, n_{ss})$ if we need to be more specific.

For example, in the case of $M_2(F)$ with idempotents e_{11} and e_{22} the trace ring will be the ring generated by all $x_{11}^{(\alpha)}$, all $x_{22}^{(\alpha)}$ and all products $x_{12}^{(\alpha)} x_{21}^{(\beta)}$. If $|X| = 1$ the Poincaré series would be

$$(1 - t(1, 1, 1))^{-1} (1 - t(2, 2, 1))^{-1} (1 - t(1, 2, 1) t(2, 1, 1))^{-1}.$$

If $(\lambda_{11}, \lambda_{22}, \lambda_{12}, \lambda_{21}) = (a, b, c, d)$ is a quadruple of one part partitions, then $m(a, b, c, d)$ is zero if $c \neq d$ and it equals 1 if $c = d$.

This is perhaps a good place to point out there is a parallel approach using symmetric group characters. For fixed $\sum n_{ij} = n$ one may consider the space V of all non-commutative polynomials multilinear in $x(i, j, \alpha)$ where for each (i, j) , α takes values from 1 to n_{ij} . Then V will be a module for the direct sum of symmetric groups $\oplus S_{n_{ij}}$. As usual, the set of identities for an algebra R will be a submodule for $\oplus S_{n_{ij}}$ and hence the quotient will also be a module. The characters of these modules would constitute the cocharacter sequence, or to be more precise, multisequence. Irreducible characters of $\oplus S_{n_{ij}}$ are indexed by tuples of partitions $\lambda_{ij} \vdash n_{ij}$. Again just as in the classical case, the multiplicity of the irreducible character $\times_{i,j} \chi^{n_{ij}}$ in the cocharacter is equal to the multiplicity $m(\lambda_{11}, \dots, \lambda_{tt})$ of the corresponding product of Schur functions in the Poincaré series. Moreover, if $e_i A e_j$ has dimension less than or equal to some d then the algebra will satisfy a Capelli identity of the form

$$\sum (-1)^\sigma (e_i x_{\sigma(1)} e_j) y_1 (e_i x_{\sigma(2)} e_j) y_2 \dots y_d e_d (e_i x_{\sigma(d+1)} e_j).$$

This will mean that $m(\langle \lambda \rangle)$ will be zero if λ_{ij} has height greater than d .

Returning to the case of $A = M_2(F)$ with two idempotents, each $e_i M_2(F) e_j$ is one dimensional so the cocharacter contains only partitions with height

less than or equal to 1. Hence, the Poincaré series in any number of variables is completely determined by the Poincaré series we already calculated in which there is one variable of each type. The general Poincaré series would be

$$\sum_{a,b,c=0}^{\infty} S_{(a)}(t(1, 1, \alpha)) S_{(b)}(t(2, 2, \beta)) S_{(c)}(t(1, 2, \gamma)) S_{(c)}(t(2, 1, \delta)).$$

Thanks to Domokos' invariant theory we are able to compute the Poincaré series in the case of pure trace identities of matrices. The main tool is Weyl's character formula which we now review.

Define the measure $d\nu(z_1, \dots, z_n)$ to be

$$(n!)^{-1} (2\pi i)^{-n} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{z_i}{z_j}\right) \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}.$$

Let M be a module for $GL_n(F)$ with character $f(z_1, \dots, z_n)$, meaning that $f(z_1, \dots, z_n)$ is the trace of a generic diagonal matrix acting on M . Then Weyl's character formula says that the dimension of the space of $GL_n(F)$ -invariants of M equals

$$\oint_T f(z_1, \dots, z_n) d\nu(z_1, \dots, z_n),$$

where T is the torus $|z_i| = 1$, for all i . A useful special case, more or less equivalent to the general case is gotten from the case in which M is the tensor product of an irreducible module with the dual of another irreducible module:

$$\oint_T S_\lambda(z_1, \dots, z_n) S_\mu(z_1^{-1}, \dots, z_n^{-1}) d\nu = \delta_{\lambda, \mu}. \quad (1)$$

For our purposes it will be better to use the following version of Weyl's character formula. It is easy to see that it is a consequence.

Theorem 3.1 (Weyl's character formula 1). *Let $M = \oplus M_\alpha$ be a graded module for $GL_n(F)$. Let $f_\alpha(z)$ be the character of M_α and let $f(z, x) = \sum_\alpha f_\alpha(z) x^\alpha$. Then the Poincaré series of the invariant space of $GL_n(F)$ equals the integral $\oint f(z, x) d\nu(z)$.*

A further generalization replaces $GL_n(F)$ with $G = GL_{n_1} \times \dots \times GL_{n_t}$. The generic diagonal matrix with entries z_1, \dots, z_n belongs to G and we continue to refer to the trace of this matrix on a module the character of the module.

Theorem 3.2 (Weyl's character formula 2). *Let $M = \oplus M_\alpha$ be a graded module for G . Let $f_\alpha(z)$ be the character of M_α and let $f(z, x) = \sum_\alpha f_\alpha(z)x^\alpha$. Then the Poincaré series of the invariant space of $GL_n(F)$ equals the integral $\oint f(z, x)d\nu(z)$, where*

$$d\nu(z) = d\nu(z_1, \dots, z_{i_1})d\nu(z_{i_1+1}, \dots, z_{i_1+i_2}) \cdots$$

In dealing with the product of general linear groups, it will be useful to denote by K the product

$$K = \prod_{1 \leq i \neq j \leq n_1} (1 - \frac{z_i}{z_j}) \prod_{n_1+1 \leq i \neq j \leq n_1+n_2} (1 - \frac{z_i}{z_j}) \times \cdots \quad (2)$$

Given $1 \leq i, j \leq n$ define $\gamma(i, j)$ to be the unique (a, b) such that the matrix unit e_{ij} belongs to $e_a M_n(F) e_b$. The polynomial ring $F[x_{ij}^{(\alpha)}]$ is a module for the group $G = GL_{n_1} \times \cdots \times GL_{n_t}$. The character is determined by the trace of a generic diagonal matrix

$$D = (\text{diag}(z_1, \dots, z_{n_1}), \text{diag}(z_{n_1+1}, \dots, z_{n_1+n_2}), \dots).$$

This character, exactly as in the classical case, is $\prod_{ij\alpha} [1 - \frac{z_i}{z_j} t(\gamma(i, j), \alpha)]^{-1}$, see [4]. Also parallel to the classical case, the Poincaré series of the fixed ring can be computed by Weyl's character formula using Theorem 2.2.

Theorem 3.3. *The Poincaré series of $U(M_n(F), ptr)$ is*

$$\oint_T \prod_{i,j,\alpha} [1 - \frac{z_i}{z_j} t(\gamma(i, j), \alpha)]^{-1} d\nu.$$

If we wish to compute the Poincaré series of the generic mixed trace ring $U(M_n(F), tr)$ using Corollary 2.3 we use the character of G acting on

$$M_n(F[x_{ij}^{(\alpha)}]) \cong M_n(F) \otimes F[x_{ij}^{(\alpha)}].$$

The character of $M_n(F)$ under conjugation by G is $\sum \frac{z_i}{z_j}$ yielding this theorem.

Theorem 3.4. *The Poincaré series of $U(M_n(F), tr)$ is*

$$\oint_T \frac{\sum z_i z_j}{\prod_{i,j,\alpha} [1 - \frac{z_i}{z_j} t(\gamma(i, j), \alpha)]} d\nu.$$

In [10] Van Den Bergh showed how to evaluate this type of integrals using graph theory. Let \mathcal{G} be the directed graph on n vertices v_i and with an edge $e(i, j, \alpha)$ from v_i to v_j for each term $(1 - \frac{z_i}{z_j} t(\gamma(i, j), \alpha))$ for $i \neq j$ in the denominator of the integrand. If C is a simple oriented cycle in \mathcal{G} we write t^C for the product of the variables corresponding to the edges in C . Here is Van Den Bergh's theorem 3.4, based on [9]:

Theorem 3.5. *The Poincaré series $P(t, ptr)$ is a rational function whose denominator can be taken to be the product of all $(1 - t^C)$, where C runs over the simple cycles of the graph. Moreover, if the variables $t(\gamma(i, j), \alpha)$ are all distinct, then this gives the least denominator.*

Van Den Bergh has an evaluation of the integral of the types in theorems 3.3 and 3.4 that works under a fairly mild connectivity hypothesis which we won't bother mentioning, but which holds in all the cases of interest. The first step in the process is to replace the integrals in theorems 3.3 and 3.4 with ones of type

$$(2\pi i)^{-n} n!^{-1} \oint_T f \prod_{ij\alpha} (1 - \frac{z_i}{z_j} t(i, j, \alpha))^{-1} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_k}{z_k}, \quad (3)$$

where in the general case f is a Laurant polynomial in z . If we can evaluate this integral we can evaluate our original one by specializing each $t(i, j, \alpha)$ to $t(\gamma(i, j), \alpha)$. Let \mathcal{G}_1 be the corresponding graph, so \mathcal{G} and \mathcal{G}_1 differ only in the labeling of the edges.

Let \mathcal{T} be a spanning tree in \mathcal{G}_1 rooted at v_1 . This means that there is a unique path in \mathcal{T} from each v_i to v_1 . For each edge in \mathcal{T} we have an equation $1 - \frac{z_i}{z_j} t(i, j, \alpha) = 0$. Together these equations let us express any homogeneous degree zero polynomial $f(z)$ in the z_i as a polynomial in the t which we write as $f|\mathcal{T}$. And given an edge e not in \mathcal{T} we write $C(e, \mathcal{T})$ for the unique cycle gotten from \mathcal{T} by adjoining e , oriented in the direction of e . Let t^C be the product of the $t(i, j, \alpha)^{\pm 1}$ corresponding to the edges in C , where the exponent will be 1 or -1 depending on whether the orientations of the corresponding edge in C and \mathcal{G}_1 agree or not. Here is Van Den Bergh's theorem:

Theorem 3.6. *The integral (3) equals the sum*

$$\sum \frac{K|\mathcal{T}}{\prod_{e \notin \mathcal{T}} (1 - t^{C(e, \mathcal{T})})}$$

where \mathcal{T} runs over all spanning trees rooted at v_1 .

The integrals from theorems [?] and [?] can be evaluated using this theorem and then specializing each $t(i, j, \alpha)$ to $t(\gamma(i, j), \alpha)$.

4 Two Examples

4.1 First Example

In our first example we take $(n+1) \times (n+1)$ matrices with idempotents $e_1 = e_{11} + \dots + e_{nn}$ and $e_2 = e_{n+1, n+1}$. We will compute the Poincaré series for the pure trace ring of the algebra generated by the generic elements $X_{12}(\alpha) = e_1 X(\alpha) e_2$ and $X_{21} = e_2 X(\alpha) e_1$ only. For ease of notation we denote $t(1, 2, \alpha) = t_\alpha$, $t(2, 1, \alpha) = u_\alpha$ and $z_{n+1} = w$. Then $K = \prod (1 - \frac{z_i}{z_j})$ where $1 \leq i \neq j \leq n$ and the Poincaré series is given by

$$(2\pi i)^{-n} n!^{-1} \oint \frac{K}{\prod_i (1 - \frac{z_i}{w} t_\alpha) (1 - \frac{w}{z_i} u_\alpha)} \frac{dw}{w} \wedge \frac{dz}{z}. \quad (4)$$

By Cauchy's theorem

$$\prod (1 - \frac{z_i}{w} t_\alpha)^{-1} = \sum S_\lambda(z_i w^{-1}) s_\lambda(t_\alpha) = \sum w^{-|\lambda|} S_\lambda(z_i) S_\lambda(t_\alpha)$$

and

$$\prod (1 - \frac{w}{z_i} u_\alpha)^{-1} = \sum S_\mu(w z_i^{-1}) S_\mu(u_\alpha) = \sum w^{|\mu|} S_\mu(z_i^{-1}) S_\mu(u_\alpha).$$

If we first integrate the product of these two sums with respect to w then, since there is no w in the factor of K we get only the terms with no w , hence

$$(2\pi i)^{-n} n!^{-1} \oint K \sum_{\lambda, \mu} S_\lambda(z_i) S_\lambda(t_\alpha) S_\mu(z_i^{-1}) S_\mu(u_\alpha) \frac{dz}{z}.$$

The integral of $K S_\lambda(z) S_\mu(z^{-1})$ equals $\delta_{\lambda, \mu}$ by equation (1). Hence the integral equals

$$\sum S_\lambda(t_\alpha) S_\lambda(u_\alpha).$$

This formula was also obtained by Bahturin and Drensky in section 3 of [1]. Note that the sum will be over only partitions with height less than or equal to n . So, if the number of t_α and u_α is each less than or equal to n then the Poincaré series will be $\prod_{\alpha, \beta} (1 - t_\alpha u_\beta)^{-1}$; and if there are more variables the Poincaré series will be a fraction of the form

$$\frac{N(t, u)}{\prod_{\alpha, \beta} (1 - t_\alpha u_\beta)}.$$

Knowledge of the numerator would be very interesting.

For the mixed trace identities in this case we would multiply the numerator of the integral in (4) by $(w + \sum z_i)(w^{-1} + \sum z_i^{-1})$. We leave the computation to the interested reader and present only the result. Given any partition λ of height less than or equal to n , let λ^+ be the set of partitions of height less than or equal to n gotten from λ by adding 1 to one of the parts. Then the Poincaré series will equal $\sum m(\lambda, \mu) S_\lambda(t_\alpha) S_\mu(u_\alpha)$ where the coefficients are given by

$$m(\lambda, \mu) = \begin{cases} 1, & \text{if } \lambda \in \mu^+ \text{ or if } \mu \in \lambda^+ \\ |\lambda^+ \cap \mu^+|, & \text{if } |\lambda| = |\mu| \\ 0, & \text{otherwise} \end{cases}$$

4.2 Second Example

For our second example we take the algebra $M_n(F)$ with idempotents $e_i = e_{ii}$ so that each $e_i M_n(F) e_j$ will be one dimensional. Hence, the Poincaré series will involve only partitions of height less than or equal to 1 and we can determine it completely by computing the integral in Theorem 3.3 with only one variable $t(i, j)$ for each i, j , i.e., by considering the Poincaré series of the trace ring of the algebra generated by the set of $t_{ij} e_{ij}$. Noting that $K = 1$ the integral is

$$(2\pi i)^{-n} \oint \prod_{ij} (1 - \frac{z_i}{z_j} t(i, j))^{-1} \frac{dz}{z}. \quad (5)$$

This integral has an interesting combinatorial interpretation. Let \mathcal{C}_n be the complete directed graph on n vertices. An \mathbb{N} -flow on \mathcal{C}_n would be an assignment of non-negative integers to each edge called the flow along that edge such that at each vertex the flow on the edges leading into that vertex equals the flow leading out. Although we cannot evaluate (5) explicitly, there are a number of things we can say. By theorem 3.6 we know that the Poincaré series will equal

$$\sum_{\mathcal{T}} \prod_{e \notin \mathcal{T}} (1 - t^{C(e, \mathcal{T})})^{-1}, \quad (6)$$

where \mathcal{T} runs over all spanning trees rooted at a given vertex v_1 . Using standard techniques we can now prove these properties of the Poincaré series.

Theorem 4.1. *Let $P(t_{11}, t_{12}, \dots, t_{nn})$ be the Poincaré series for the trace ring, T , of the ring generated by the set $\{x_{ij} e_{ij}\}$, $i, j = 1, \dots, n$.*

1. For each cycle $\sigma = (i_1, i_2, \dots, i_a) \in S_n$ we write $1 - t_\sigma = 1 - \prod_{b=1}^a t_{i_b, i_{b+1}}$ with the convention that $i_{a+1} = i_1$. Then $P(t_{11}, t_{12}, \dots, t_{nn})$ is a rational function with least denominator the product of all such $1 - t_\sigma$, $\sigma \in S_n$.
2. The Poincaré series will satisfy a functional equation

$$P(t_{11}^{-1}, \dots, t_{nn}^{-1}) = -P(t_{11}, \dots, t_{nn}) \prod_{ij} t_{ij}.$$

3. If we specialize each t_{ij} to a new variable t , for each $i \neq j$ then the resulting function has a pole at $t = 1$ of order $n^2 - 1$.

Proof. The first part is an immediate consequence of Theorem 3.5.

For the second part, It is an interesting exercise which we leave to the reader, to show that in (6) for each \mathcal{T} that

$$\prod_{e \notin \mathcal{T}} (1 - t^{-C(e, \mathcal{T})})^{-1} = - \prod_{ij} t_{ij} \prod_{e \notin \mathcal{T}} (1 - t^{C(e, \mathcal{T})})^{-1}.$$

The third part follows from an argument in [4]. The trace ring T can be written as a tensor product $F[x_{11}, \dots, x_{nn}] \otimes T'$, where T' is the trace ring of the algebra generated by the $x_{ij}e_{ij}$, with $i \neq j$, and so Poincaré series can be written as $\prod_i (1 - t_{ii})^{-1} P(T')$. Let $\langle X \rangle$ be the free abelian group generated by the x_{ij} with $i \neq j$, let $\langle Z \rangle$ be the free abelian group generated by z_1, \dots, z_n , and let $\langle U \rangle$ be the free abelian group generated by u ; and consider the exact sequence

$$1 \longrightarrow K \longrightarrow \langle X \rangle \xrightarrow{f} \langle Z \rangle \xrightarrow{g} \langle U \rangle \longrightarrow 1,$$

where $f(x_{ij}) = z_i z_j^{-1}$ and $g(z_i) = u$, for all i, j . It follows that the rank of K is $n^2 - n + 1$. But $F(K)$ is the quotient field of the trace ring T' is and so T will have GK-dimension $n^2 - 1$ and the theorem follows. \square

Using the notation P_n to display the dependence on n , here are the first few:

$$\begin{aligned} P_1 &= \frac{1}{(1 - t_{11})} \\ P_2 &= \frac{1}{(1 - t_{11})(1 - t_{22})(1 - t_{12}t_{21})} \\ P_3 &= \frac{1 - t_{12}t_{13}t_{23}t_{32}t_{31}t_{21}}{\prod_i (1 - t_{ii})(1 - t_{12}t_{21})(1 - t_{13}t_{31})(1 - t_{23}t_{32})(1 - t_{12}t_{23}t_{31})(1 - t_{13}t_{32}t_{23})} \end{aligned}$$

For the mixed traces, let Poincaré series we would multiply the integrand in (5) by $\sum \frac{z_i}{z_j}$. Call the Poincaré series be Q_n . Then Q_n is a rational function with the same denominator as P_n and, just like P_n , if we specialize each t_{ij} with $i \neq j$ to t in Q_n we get a pole of order $n^2 - 1$ in the resulting fraction. However, Q_n does not satisfy a functional equation as in Theorem 4.1 .2. Then $P_1 = Q_1$ and $Q_2 =$

$$\frac{2 + t_{12} + t_{21}}{(1 - t_{11})(1 - t_{22})(1 - t_{12}t_{21})}.$$

The numerator of Q_3 is

$$\begin{aligned} & -3t_{1,2}t_{1,3}t_{2,1}t_{2,3}t_{3,1}t_{3,2} - t_{1,3}t_{2,1}t_{2,3}t_{3,1}t_{3,2} - t_{1,2}t_{2,1}t_{2,3}t_{3,1}t_{3,2} \\ & - t_{1,2}t_{1,3}t_{2,3}t_{3,1}t_{3,2} - t_{1,2}t_{2,3}t_{3,1}t_{3,2} - t_{1,2}t_{1,3}t_{2,1}t_{3,1}t_{3,2} \\ & - t_{1,3}t_{2,1}t_{3,1}t_{3,2} - t_{1,2}t_{1,3}t_{2,1}t_{2,3}t_{3,2} - t_{1,3}t_{2,1}t_{2,3}t_{3,2} \\ & - t_{1,2}t_{1,3}t_{2,1}t_{3,2} + t_{2,1}t_{3,2} + t_{1,3}t_{3,2} + t_{3,2} - t_{1,2}t_{1,3}t_{2,1}t_{2,3}t_{3,1} \\ & - t_{1,2}t_{2,1}t_{2,3}t_{3,1} - t_{1,2}t_{1,3}t_{2,3}t_{3,1} + t_{2,3}t_{3,1} + t_{1,2}t_{3,1} + t_{3,1} \\ & + t_{1,2}t_{2,3} + t_{2,3} + t_{1,3}t_{2,1} + t_{2,1} + t_{1,3} + t_{1,2} + 3 \end{aligned}$$

5 A-identities

Regev's theorem concerns a double centralizer between the wreath product $A \sim S_n$ where $A \subseteq \text{End}(V)$ is a semisimple algebra and the group \mathcal{G} of units in $\text{End}_A(V)$. Domokos limited his attention to the case in which the action is multiplicity free and $A = F^t$. This leads to the very natural theory of E -trace identities. It is possible to reproduce many of Domokos' results in a more general setting. Let $F\langle X, A \rangle$ be the amalgamated product $A * F\langle X \rangle$. Elements of $F\langle X, A \rangle$ are linear combinations of terms of the form $a_1x_{i_1}a_2 \cdots a_mx_{i_m}a_{m+1}$. We will call these A -polynomials. It makes sense to talk about A -identities for any A -algebra, in particular for $M_n(F) = \text{End}(V)$. Since the a_i are in $M_n(F)$ it makes sense to substitute arbitrary matrices for the x_i and to talk about A -identities for $M_n(F)$. Likewise we can define A -trace polynomials, either mixed or pure.

Definition 5.1. Given $\sigma \in S_k$ whose inverse has cycle decomposition

$$\sigma^{-1} = (i_1, \dots, i_a)(j_1, \dots, j_b) \cdots$$

we define the trace monomial tr_σ via

$$tr_\sigma(x_1, \dots, x_k) = tr(x_{i_1} \cdots x_{i_a})tr(x_{j_1} \cdots x_{j_b}) \cdots$$

Given $w = (a_1 \otimes \cdots \otimes a_k, \sigma)$ a monomial in $A \sim S_k$ we define $tr_w(x_1, \dots, x_k)$ to be $tr_\sigma(x_1 a_1, \dots, x_k a_k)$. More generally, if $w = \sum w_i \in W_k$ is a sum of monomials we let tr_w be $\sum tr_{w_i}$.

Pure A -trace polynomials are important for invariant theory:

Theorem 5.2. *The A -pure trace polynomials give all function $f : M_n(F)^k \rightarrow F$ invariant under simultaneous conjugation from \mathcal{G} .*

Proof. By multilinearization we need only consider the multilinear maps. Identifying $M_n(F)$ with $End(V)$ for an n -dimensional vector space V , we consider $[(End(V)^{\otimes k})^*]^{\mathcal{G}}$. Now, as G -modules

$$\begin{aligned} (End(V)^{\otimes k})^* &\cong \left((V \otimes V^*)^{\otimes k} \right)^* \\ &\cong (V^{\otimes k} \otimes V^{*\otimes k})^* \\ &\cong End(V^{\otimes k}). \end{aligned}$$

The last isomorphism requires comment. Let $T \in End(V^{\otimes k})$. We associate to T the functional on $V^{\otimes k} \otimes V^{*\otimes k}$ which takes $\underline{v} \otimes \underline{\phi}$ to $\underline{\phi}(T(\underline{v}))$. Hence, the \mathcal{G} invariants come from $End_{\mathcal{G}}(V^{\otimes k})$ which equals $A \sim S_k$.

To complete the proof let $x_i = v_i \otimes \phi_i \in V \otimes V^* \equiv End(V)$, let $(a_1 \otimes \cdots \otimes a_k)\sigma \in A \sim S_k$ and compute

$$\begin{aligned} \underline{\phi}((a_1 \otimes \cdots \otimes a_k)\sigma(\underline{v})) &= \underline{\phi}(a_1 v_{\sigma^{-1}(1)} \otimes \cdots \otimes a_k v_{\sigma^{-1}(k)}) \\ &= \phi_1((a_1 v_{\sigma^{-1}(1)}) \cdots \phi_k(a_k v_{\sigma^{-1}(k)}) \\ &= \phi_1 \circ a_1(v_{\sigma^{-1}(1)}) \cdots \phi_k \circ a_k(v_{\sigma^{-1}(k)}) \\ &= tr_\sigma(v_1 \otimes \phi_1 a_1, \dots, v_k \otimes \phi_k a_k) \\ &= tr_\sigma(x_1 a_1, \dots, x_k a_k) \\ &= tr_{(a_1 \otimes \cdots \otimes a_k)\sigma}(x_1, \dots, x_k) \end{aligned}$$

and this completes the proof. \square

As a corollary to the proof we have

Corollary 5.3. *A multilinear pure A -polynomial tr_w is an A -identity for $E = End(V)$ if and only if w is in the kernel of the map $A \sim S_k \rightarrow End(V^{\otimes k})$.*

Theorem 5.2 can be restated in terms of generic matrices. Let $R(A, n, k)$ be the ring generated by the generic $n \times n$ matrices X_1, \dots, X_k , $X_\alpha = (x_{ij}^{(\alpha)})$,

together with $A \subseteq M_n(F)$; and let $\bar{C}(A, n, k)$ be the ring generated by the traces of elements of $R(A, n, k)$. The group \mathcal{G} acts on the polynomial ring $F[x_{ij}^{(\alpha)}]$ via by defining $g(x_{ij}^{(\alpha)})$ to be the (i, j) entry of $gX_\alpha g^{-1}$. Just as in the classical case Theorem 5.2 is equivalent to the following.

Theorem 5.4. *The fixed ring $F[x_{ij}^{(\alpha)}]^\mathcal{G}$ equals the generic trace ring $\bar{C}(A, n, k)$.*

Since A is a semisimple algebra it can be written as a direct sum of simple algebras $A = \sum_{i=1}^t A_i$. Let e_i be a two sided idempotent in A_i and assume $A_i \cong M_{n_i}(F)$. The pure trace identities of $M_n(F)$ are generated by the Cayley-Hamilton polynomial

$$C_{n+1}(x_1, \dots, x_{n+1}) = \sum_{\sigma \in S_{n+1}} (-1)^\sigma \text{tr}_\sigma(x_1, \dots, x_{n+1}).$$

In our case A will satisfy the identities

$$C_{n_i+1}(x_1 e_i, \dots, x_{n_i+1} e_i). \quad (7)$$

Similar to both the classical case and Domokos' generalization we have the following theorem

Theorem 5.5. *All pure A -trace identities for $M_n(F)$ are consequences of those in (7), where $i = 1, \dots, t$.*

Since it is an oft told tale at this point, we merely sketch the proof. By a multilinearization argument, it suffices to prove the theorem for multilinear identities. As in the classical case we have the following lemma.

Lemma 5.6. *Given $w \in A \sim S_k$, $\sigma \in S_k$ and invertible $b = B_1 \otimes \dots \otimes B_k \in A^{\otimes k}$,*

$$\text{tr}_{\sigma w \sigma^{-1}}(x_1, \dots, x_k) = \text{tr}_w(x_{\sigma(1)}, \dots, x_{\sigma(k)})$$

and

$$\text{tr}_{b w b^{-1}}(x_1, \dots, x_k) = \text{tr}_w(B_1^{-1} x_1 B_1, \dots, B_k^{-1} x_k B_k).$$

Given an algebra R with A -action and trace, let W_k be the set of $w \in A \sim S_k$ such that tr_w is an identity for R . The lemma shows that W_k is closed under conjugation, but it may not be closed under right or left multiplication. We use it to prove the following.

Lemma 5.7. *Assume that W_k is a two-sided ideal of $A \sim S_k$. Then for all $m \geq k$ the elements of $(A \sim S_m)W_k(A \sim S_m)$ are consequences of W_k and hence identities for R .*

Sketch of Proof. Using induction it suffices to do the case of $m = k + 1$. Using the previous lemma it suffices to consider $(A \sim S_m)W_k$. And, since W_k is closed under multiplication from $A \sim S_k$, we reduce to the product $(a_1 \otimes \cdots \otimes a_{k+1}\sigma)u$ where $u \in W_k$ and either σ is the identity or a transposition of the form $(i, k + 1)$. In the case of σ equal to the identity it is not hard to prove that

$$tr_{a_1 \otimes \cdots \otimes a_{k+1}u}(x_1, \dots, x_{k+1}) = tr_{a_1 \otimes \cdots \otimes a_k u}(x_1, \dots, x_k) tr(x_{k+1} a_{k+1}).$$

Since W_k is closed under multiplication by $A^{\otimes k}$, this is certainly a consequence of an element of W_k .

Finally, in the case of $\sigma = (i, k + 1)$ we re-write $(a_1 \otimes \cdots \otimes a_{k+1})\sigma$ as $\sigma(b_1 \otimes \cdots \otimes b_{k+1})$ and, again use fact that W_k is closed under multiplication from $A^{\otimes k}$ to reduce to $(i, k + 1)(1 \otimes \cdots \otimes 1 \otimes b_{k+1})u$. We leave it to the reader to prove in this case that

$$tr_{(i,k+1)(1 \otimes \cdots \otimes 1 \otimes b_{k+1})u}(x_1, \dots, x_{k+1}) = tr_u(x_1, \dots, x_k)|_{x_i=x_{k+1}b_{k+1}x_i}.$$

□

By Corollary 5.3 W_k , the identities of $M_n(F)$ in $A \sim S_k$ will be a two-sided ideal, identified with the kernel of the map $A \sim S_k \rightarrow End(V)$. The two sided ideals $I_{\langle \lambda \rangle}$ $A \sim S_k$ are parameterized by t -tuples of partitions $\langle \lambda \rangle = (\lambda_1, \dots, \lambda_t)$ with $\sum |\lambda_i| = k$. The proof of the following is the same as the proof of Theorem 2.1(ii) in [3].

Theorem 5.8. *The kernel of the map $A \sim S_k \rightarrow End(V^{\otimes k})$ equals the sum of all ideals $I_{\langle \lambda \rangle}$ with some λ_i of height greater than n_i .*

In light of Lemma 5.7 we need to see how ideals of $A \sim S_k$ induce up to $A \sim S_m$. As Domokos noted, the next lemma follows from the Appendix of [5].

Lemma 5.9 (The Branching Lemma). *The ideal $I_{\langle \lambda \rangle} \trianglelefteq (A \sim S_k)$ induces up to $A \sim S_m$, $m > k$ as $\oplus I_{\langle \mu \rangle}$ summed over all $\langle \lambda \rangle \subseteq \langle \mu \rangle$, i.e., $\lambda_i \subseteq \mu_i$ for each i .*

We now have all the ingredients we need to prove Theorem 5.5.

Proof of Theorem 5.5. Identifying $A \sim S_m$ with multilinear pure A -trace polynomials as in Definition 5.1, Corollary 5.3 says that the identities of $M_n(F)$ equals the kernel of the map $A \sim S_m \rightarrow End(V^{\otimes m})$. Next, Theorem 5.8 identifies that kernel as the sum of the ideals $I_{\langle \mu \rangle}$ with some μ_i of

height greater than $n_i + 1$. A partition μ_i will have height greater than or equal to $n_i + 1$ if and only if it contains the partition (1^{n_i+1}) . By Lemma 5.9 the ideals $I_{\langle\mu\rangle}$ are induced from the ideals $I_{\lambda(i)}$ where $\lambda(i) = (\lambda_1, \dots, \lambda_i)$ and

$$\lambda_j = \begin{cases} \text{the empty partition} & \text{if } j \neq i \\ (1^{n_i+1}) & \text{if } j = i. \end{cases}$$

By Lemma 5.7 all identities of $M_n(F)$ are consequences of those in $I_{\lambda(i)}$. These ideals are one dimensional and correspond to the polynomials in (7). \square

The results of this section all have analogues for mixed trace identities and are easily derived from them using the facts that $f(x_1, \dots, x_k)$ is a mixed trace identity if and only if $tr(f(x_1, \dots, x_k)x_{k+1})$ is a pure trace identity, and every pure trace identity $g(x_1, \dots, x_{k+1})$ can be written in the form $tr(f(x_1, \dots, x_k)x_{k+1})$ for a pure trace identity f . Since the techniques are standard we merely state the results.

Theorem 5.10. *The A -mixed trace polynomials give all function $f : M_n(F)^k \rightarrow M_n(F)$ invariant under simultaneous conjugation from \mathcal{G} .*

Let $\bar{R}(A, n, k)$ be the algebra generated by $R(A, n, k)$ and $\bar{C}(A, n, k)$, which means that it is the algebra with trace generated by the generic matrices X_1, \dots, X_k and by A .

Theorem 5.11. *The fixed ring $M_n(F[x_{ij}^{(\alpha)}])^{\mathcal{G}}$ equals the ring $\bar{R}(A, n, k)$.*

Define the mixed trace Cayley-Hamilton polynomials via

$$tr(MC_n(x_1, \dots, x_n)x_{n+1}) = C_{n+1}(x_1, \dots, x_{n+1})$$

Theorem 5.12. *All mixed A -trace identities for $M_n(F)$ are consequences of the identities*

$$MC_{n_i}(e_i x_i e_i, \dots, i_i x_{n_i} e_i).$$

6 Poincaré Series Again

We consider the case of $G = GL_a(F) = GL(W)$ acting on $V = W^b$ which we write as $W_1 \oplus \dots \oplus W_b$. Let $n = ab$. The commuting ring $End_G(V)$ is isomorphic to $M_b(F)$ with the matrix unit e_{ij} an isomorphism $W_i \rightarrow W_j$. For example, if $a = b = 2$ then

$$e_{11} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_{12} \leftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{etc.}$$

Write the matrix unit in $M_{ab}(F)$ corresponding to e_{ij} in $M_a(F)$ as E_{ij} . As in the classical case, the universal algebra for algebras satisfying the G -identities of $M_{ab}(F)$ is the algebra generated by all $E_i X_\alpha E_j$ where the X_α are generic $n \times n$ matrices. Let $U(a, b, k)$ be this universal algebra on k generators; let $U(a, b, k, ptr)$ be the algebra generated by the traces; and let $U(a, b, k, tr)$ be the algebra generated by both. Theorems 5.2 and ?? imply the following.

Theorem 6.1. *$U(a, b, k, ptr)$ is the fixed ring of G acting on the polynomial ring $F[x_{ij}^{(\alpha)}]$ and $U(a, b, k, tr)$ is the fixed ring of G acting on the matrix ring over the polynomial algebra $M_n(F[x_{ij}^{(\alpha)}])$.*

Using Weyl's integration formula we can express the Poincaré series for the pure and mixed trace rings as complex integrals. Let $D \in GL_a(F)$ be the generic diagonal matrix $diag(z_1, \dots, z_a)$. Then D acts on W as

$$D^{\oplus b} = diag(\underbrace{z_1, \dots, z_1}_b, \dots, \underbrace{z_a, \dots, z_a}_b).$$

Given $1 \leq i, j \leq n$ we let $\gamma(i, j)$ be the unique (s, t) such that $e_{ij} \in E_s M_n(F) E_t$. We also let $s = \gamma_1(i, j)$ and $t = \gamma_2(i, j)$. Hence, under the conjugation action e_{ij} would be an eigenvector for D with eigenvalue $z_s z_t^{-1}$. If we grade $F[x_{ij}^{(\alpha)}]$ by \mathbb{N}^{ka^2} as in section 3, then $U(a, b, k, ptr)$ has Poincaré series

$$(2\pi i)^{-a} a!^{-1} \oint_T \frac{\prod_{i \neq j} (1 - \frac{z_i}{z_j})}{\prod_{ij\alpha} (1 - \frac{\gamma_1(i,j)}{\gamma_2(i,j)} t(i, j, \alpha))^b} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_a}{z_a} \quad (8)$$

Let $P(a, b, k)$ denote this integral. In the special case of $b = 1$ we are reduced to the classical case of $GL_a(F)$ acting on $a \times a$ matrices and the Poincaré series is given by the integral in theorem 3.3. Comparing the integrands of that theorem to (8) we have the following.

Theorem 6.2. *Let $P(a, 1, bk)$ be a function of the variables $t(i, j, (m, \alpha))$, where $1 \leq i, j \leq a$, $1 \leq m \leq b$ and $1 \leq \alpha \leq k$; and let $P(a, b, k)$ be a function of the variables $t(i, j, \alpha)$, where $1 \leq i, j \leq a$ and $1 \leq \alpha \leq k$. Then $P(a, b, k)$ can be computed from $P(a, 1, bk)$ by taking the limit of $t(i, j, (m, \alpha)) \rightarrow t(i, j, \alpha)$ for each variable.*

We can also restate Theorem 6.2 in terms of cocharacters. We define an operation on Schur functions via

$$S_\lambda^{\boxtimes b}(t_1, \dots, t_a) = S_\lambda(\underbrace{t_1, \dots, t_1}_b, \dots, \underbrace{t_a, \dots, t_a}_b).$$

This is equivalent to

$$S_{\lambda}^{\boxtimes b}(t_1, \dots, t_a) = \sum_{\mu_1 \subseteq \mu_2 \subseteq \dots \subseteq \mu_{b-1} \subseteq \lambda} S_{\mu_1}(t_a, \dots, t_a) S_{\mu_2/\mu_1}(t_a, \dots, t_a) \cdots S_{\lambda/\mu_{b-1}}(t_a, \dots, t_a).$$

We may define a corresponding operation on S_n -characters via

$$(x^{\lambda})^{\boxtimes b} = \sum_{\mu_1 \subseteq \mu_2 \subseteq \dots \subseteq \mu_{b-1} \subseteq \lambda} \chi^{\mu_1} \otimes \chi^{\mu_2/\mu_1} \otimes \dots \otimes \chi^{\lambda/\mu_{b-1}},$$

where the tensor product is the outer tensor product.

Theorem 6.3. *Let the ring of generic $a \times a$ matrices have Poincaré series $\sum m_{\lambda} S_{\lambda}(t_1, \dots, t_k)$. Then $U(a, b, k)$ has Poincaré series $\sum m_{\lambda} S_{\lambda}^{\boxtimes b}(t_1, \dots, t_k)$. And if $M_a(F)$ has m -th cocharacter $\sum m_{\lambda} \chi^{\lambda}$, then $M_{ab}(F)$ will have m -th G -cocharacter $\sum m_{\lambda} (\chi^{\lambda})^{\boxtimes b}$*

7 Procesi's Embedding Theorem

In [7] Procesi proved the following theorem:

Theorem (Procesi). *Let R be an algebra with trace in characteristic zero, satisfying all of the trace identities of $M_n(F)$. Then R has a trace preserving embedding into $M_n(C)$ for some commutative algebra C .*

As Procesi carefully points out, his proof follows from the following:

- The free ring with trace $F\langle X, tr \rangle$ modulo the trace identities of $M_n(F)$ is isomorphic to the generic matrix algebra with trace.
- This free ring is the fixed ring of $G = GL_n(F)$ acting on $M_n(F[x_{ij}^{(\alpha)}])$.
- The group $G = GL_n(F)$ is linearly reductive.

Using Corollary 2.4 and Theorem 2.6 This immediately gives the following:

Theorem 7.1. *Let G be a linearly reductive subgroup of $GL_n(F)$ and let $A = \text{End}_G(F^n) \subseteq M_n(F)$. If R is an A -algebra with trace satisfying all of the A -trace identities of $M_n(F)$, then R has an embedding into $M_n(C)$ for some commutative algebra C which preserves both trace and A -action.*

For example, take $G = GL_{n_1} \times \dots \times GL_{n_t} \subseteq GL_n$, where $n = \sum n_i$. Then $A = F^t$ and an A -algebra is what we called an E -algebra.

Corollary 7.2. *Let R be an E -algebra satisfying the multilinear Cayley-Hamilton identities*

$$CH_{n_i+1}(E_i X_1 E_i, \dots, E_i X_{n_i+1} E_i)$$

as in Theorem 2.5. Then R has an E -trace preserving embedding into $M_n(C)$ for some commutative algebra C .

References

- [1] Yu. A. Bahturin and V. Drensky, Identities of bilinear mappings and graded polynomial identities of matrices, *Lin. Alg. and Appl.* **369** (2003), 95–112.
- [2] A. Berele and A. Regev. Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. in Math.* **64** (1987), 118–175.
- [3] Domokos, Invariants of quivers and wreath products, *Comm. in Alg.* **26** (2007), 2807–2819.
- [4] E. Formanek, Invariants and the ring of generic matrices, *J. Algebra* **89** (1984), 178–223.
- [5] A. Giambruno and A. Regev, Wreath products and P.I. algebras, *J. Pure Appl. Algebra* **35** (1985), 133–149.
- [6] A. Giambruno and M. Zaicev, *Polynomial Identities and Asymptotic Methods*, Mathematical Surveys and Monographs, vol. 122, American Mathematical Society, Providence, RI, 2005.
- [7] C. Procesi, A formal inverse to the Cayley-Hamilton theorem, *J. Alg.* **107** (1987), 63–74.
- [8] A. Regev, The representations of wreath products via double centralizing theorems, *J. Alg.* **102** (1986), 423–443.
- [9] R. Stanley, Linear homogeneous equations and magic labellings of graphs, *Duke Math. Journal* **40** (1973), 607–632.
- [10] M. Van Den Bergh, Explicit rational forms for the Poincaré series of the trace ring of generic matrices, *Israel J. Math.* **73** (1991), 17–31.